

# Estimates for approximations by Fourier sums, best approximations and best orthogonal trigonometric approximations of the classes of $(\psi, \beta)$ -differentiable functions

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## Abstract

We obtain the exact-order estimates for approximations by Fourier sums, best approximations and best orthogonal trigonometric approximations in metrics of spaces  $L_s$ ,  $1 \leq s < \infty$ , of classes of  $2\pi$ -periodic functions, whose  $(\psi, \beta)$ -derivatives belong to unit ball of the space  $L_\infty$ .

We denote by  $L_p$ ,  $1 \leq p < \infty$ , the space of  $2\pi$ -periodic functions  $f : \mathbb{R} \rightarrow \mathbb{C}$ , summable to the power  $p$  on  $[0, 2\pi)$ , in which the norm is given by the formula  $\|f\|_p = \left( \int_0^{2\pi} |f(t)|^p dt \right)^{\frac{1}{p}}$ ; and we denote by  $L_\infty$  the space of  $2\pi$ -periodic measurable and essentially bounded functions  $f : \mathbb{R} \rightarrow \mathbb{C}$  with the norm  $\|f\|_\infty = \operatorname{ess\,sup}_t |f(t)|$ ;

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the function from  $L_1$ , whose Fourier series has the form

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx},$$

where  $\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt$  are Fourier coefficients of the function  $f$ ,  $\psi(k)$  is an arbitrary fixed sequence of real numbers and  $\beta$  is a fixed real number. Then, if the series

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\hat{f}(k)}{\psi(|k|)} e^{i(kx + \frac{\beta\pi}{2} \operatorname{sign} k)}$$

is the Fourier series of some function  $\varphi$  from  $L_1$ , then this function is called the  $(\psi, \beta)$ -derivative of the function  $f$  and denoted by  $f_\beta^\psi$ . A set of functions  $f$ , whose  $(\psi, \beta)$ -derivatives exist is denoted by  $L_\beta^\psi$  (see [1]).

If  $f \in L_\beta^\psi$  and, at the same time,  $f_\beta^\psi \in \mathfrak{N}$ , where  $\mathfrak{N} \subseteq L_1$ , then we say that the function  $f$  belongs to the class  $L_\beta^\psi \mathfrak{N}$ . By  $B_{R,p}$  we denote the balls of the radius  $R$  of real-valued functions from  $L_p$ , i.e., the sets

$$B_{R,p} := \{\varphi : \mathbb{R} \rightarrow \mathbb{R}, \|\varphi\|_p \leq R\}, \quad R > 0, \quad 1 \leq p \leq \infty.$$

In present paper as  $\mathfrak{N}$  we take the unit balls  $B_{1,p}$ . Herewith, the functional classes  $L_{\beta}^{\psi} B_{1,p}$  are denoted by  $L_{\beta,p}^{\psi}$ .

In the case  $\psi(k) = k^{-r}$ ,  $r > 0$ , the classes  $L_{\beta,p}^{\psi}$  are well-known Weyl–Nagy classes  $W_{\beta,p}^r$ .

For functions  $f$  from classes  $L_{\beta,p}^{\psi}$  we consider:  $L_s$ -norms of deviations of the functions  $f$  from their partial Fourier sums of order  $n-1$ , i.e., the quantities

$$\|\rho_n(f; \cdot)\|_s = \|f(\cdot) - S_{n-1}(f; \cdot)\|_s, \quad 1 \leq s \leq \infty, \quad (1)$$

where

$$S_{n-1}(f; x) = \sum_{k=-n+1}^{n-1} \hat{f}(k) e^{ikx};$$

best orthogonal trigonometric approximations of the functions  $f$  in metric of space  $L_s$ , i.e., the quantities of the form

$$e_m^{\perp}(f)_s = \inf_{\gamma_m} \|f(\cdot) - S_{\gamma_m}(f; \cdot)\|_s, \quad 1 \leq s \leq \infty, \quad (2)$$

where  $\gamma_m$ ,  $m \in \mathbb{N}$ , is an arbitrary collection of  $m$  integer numbers, and

$$S_{\gamma_m}(f; x) = \sum_{k \in \gamma_m} \hat{f}(k) e^{ikx};$$

and best approximations of the functions  $f$  in space  $L_s$ , i.e., the quantities of the form

$$E_n(f)_s = \inf_{t_{n-1} \in \mathcal{T}_{2n-1}} \|f - t_{n-1}\|_s, \quad 1 \leq s \leq \infty, \quad (3)$$

where  $\mathcal{T}_{2n-1}$  is the subspace of all trigonometric polynomials  $t_{n-1}$  with real coefficients of degrees not greater than  $n-1$ .

We set

$$\mathcal{E}_n(L_{\beta,p}^{\psi})_s = \sup_{f \in L_{\beta,p}^{\psi}} \|\rho_n(f; \cdot)\|_s, \quad 1 \leq p, s \leq \infty, \quad (4)$$

$$e_n^{\perp}(L_{\beta,p}^{\psi})_s = \sup_{f \in L_{\beta,p}^{\psi}} e_n^{\perp}(f)_s, \quad 1 \leq p, s \leq \infty, \quad (5)$$

$$E_n(L_{\beta,p}^{\psi})_s = \sup_{f \in L_{\beta,p}^{\psi}} E_n(f)_s, \quad 1 \leq p, s \leq \infty. \quad (6)$$

The following inequalities follow from given above definitions (4)–(6)

$$E_n(L_{\beta,p}^{\psi})_s \leq \mathcal{E}_n(L_{\beta,p}^{\psi})_s, \quad e_{2n-1}^{\perp}(L_{\beta,p}^{\psi})_s \leq \mathcal{E}_n(L_{\beta,p}^{\psi})_s, \quad 1 \leq p, s \leq \infty. \quad (7)$$

In present paper we solve the problem about finding the exact order estimates of the quantities  $\mathcal{E}_n(L_{\beta,\infty}^{\psi})_s$ ,  $E_n(L_{\beta,\infty}^{\psi})_s$  and  $e_n^{\perp}(L_{\beta,\infty}^{\psi})_s$  for  $1 \leq s < \infty$ ,  $\beta \in \mathbb{R}$ .

For the Weyl–Nagy classes the exact order estimates of the quantities  $\mathcal{E}_n(W_{\beta,p}^r)_s$  and  $E_n(W_{\beta,p}^r)_s$  are known for all admissible values of parameters  $r$ ,  $p$ ,  $s$  and  $\beta$ , i.e., for  $r > \max\{\frac{1}{p} - \frac{1}{s}, 0\}$ ,  $\beta \in \mathbb{R}$  and  $1 \leq p, s \leq \infty$  (see, e.g., [2, p. 47–49]). What concerning the

best orthogonal trigonometric approximations  $e_n^\perp(W_{\beta,p}^r)_s$ , so order estimates are known for them (see [3]–[9]) for various (but not for all possible) values of the parameters  $r, p, s$  and  $\beta$ .

Order estimates of the quantities (4)–(6) under certain restrictions for the parameters  $r, p, s$  and  $\beta$  were established in the works [1], [10]–[20]. However, the case  $p = \infty$ ,  $1 \leq s \leq \infty$  for some or another reasons hasn't been investigated yet.

We denote by  $P$  the set of positive, almost decreasing sequences  $\psi(k)$ ,  $k \geq 1$ , (we remind, that sequence  $\psi(k)$  almost decreases, if there exists a positive constant  $M$  such that for arbitrary  $k_1 \leq k_2$  the following inequality is satisfied  $\psi(k_2) \leq M\psi(k_1)$ ) such that

$$\sup_{m \in \mathbb{N}} \sum_{k=2^m}^{2^{m+1}} |\psi_n(k+1) - \psi_n(k)| \leq K\psi(n),$$

where

$$\psi_n(k) = \begin{cases} 0, & k < n, \\ \psi(k), & k \geq n, \end{cases}$$

and  $K$  is the quantity uniformly bounded with respect to  $n$ .

**Theorem 1.** *Let  $\psi \in P$ ,  $1 \leq s < \infty$  and  $\beta \in \mathbb{R}$ . Then*

$$E_n(L_{\beta,\infty}^\psi)_s \asymp \mathcal{E}_n(L_{\beta,\infty}^\psi)_s \asymp \psi(n). \quad (8)$$

Here and in what follows, we write  $A(n) \asymp B(n)$  for positive sequences  $A(n)$  and  $B(n)$  to denote that there are positive constants  $K_1$  and  $K_2$  such that  $K_1 B(n) \leq A(n) \leq K_2 B(n)$ ,  $n \in \mathbb{N}$ .

*Proof.* At first let's prove that the following inequality is true

$$\mathcal{E}_n(L_{\beta,\infty}^\psi)_s \leq K^{(1)}\psi(n), \quad 1 \leq s < \infty. \quad (9)$$

In inequality (9) and henceforth by  $K^{(i)}$ ,  $i = 1, 2, \dots$  we denote quantities uniformly bounded with respect to  $n$ .

If  $f \in L_{\beta,\infty}^\psi$ , then

$$\|f_\beta^\psi\|_s \leq (2\pi)^{\frac{1}{s}} \|f_\beta^\psi\|_\infty \leq (2\pi)^{\frac{1}{s}}, \quad (10)$$

and so, it is obviously that

$$L_{\beta,\infty}^\psi \subset L_\beta^\psi B_{(2\pi)^{\frac{1}{s}},s} \subset L_\beta^\psi L_s, \quad 1 \leq s < \infty. \quad (11)$$

The following proposition follows from the theorem 6.7.1 in [1].

**Proposition 1.** *Let  $1 < s < \infty$ ,  $\psi \in P$ ,  $f \in L_\beta^\psi L_s$  and  $\beta \in \mathbb{R}$ . Then for arbitrary  $n \in \mathbb{N}$  there exists a positive constant  $K$ , which is uniformly bounded with respect to  $n$  and  $f$  and such that*

$$\|\rho_n(f; x)\|_s \leq K\psi(n)E_n(f_\beta^\psi)_s. \quad (12)$$

Taking into account (10), (11) and in view of proposition 1, we obtain the following estimates

$$\mathcal{E}_n(L_{\beta,\infty}^\psi)_s \leq \mathcal{E}_n(L_\beta^\psi B_{(2\pi)^{\frac{1}{s}},s})_s \leq (2\pi)^{\frac{1}{s}} K\psi(n), \quad 1 < s < \infty. \quad (13)$$

Thus, the inequalities (9) are proved for  $1 < s < \infty$ .

Let's show the rightness of correlation (9) for  $s = 1$ . We use the following statement (see, e.g., [2, p. 8]).

**Proposition 2.** *Let  $1 \leq q \leq p \leq \infty$ . On this if  $f \in L_p$ , then  $f \in L_q$  and*

$$\|f\|_q \leq (2\pi)^{\frac{1}{q}-\frac{1}{p}} \|f\|_p. \quad (14)$$

By using (14) for  $q = 1$ ,  $p = 2$  and inequality (13) for  $s = 2$ , we obtain

$$\begin{aligned} \mathcal{E}_n(L_{\beta,\infty}^\psi)_1 &= \sup_{f \in L_{\beta,\infty}^\psi} \|f(\cdot) - S_{n-1}(f; \cdot)\|_1 \leq \\ &\leq (2\pi)^{\frac{1}{2}} \sup_{f \in L_{\beta,\infty}^\psi} \|f(\cdot) - S_{n-1}(f; \cdot)\|_2 = (2\pi)^{\frac{1}{2}} \mathcal{E}_n(L_{\beta,\infty}^\psi)_2 \leq K^{(1)}\psi(n). \end{aligned} \quad (15)$$

The rightness of the inequality (9) follows from (13) and (15).

To obtain the lower bound of the quantity  $E_n(L_{\beta,\infty}^\psi)_s$ , we consider the following function

$$f_1(t) = f_1(\psi; n; t) = \psi(n) \cos nt.$$

It is obviously, that  $f_1 \in L_{\beta,\infty}^\psi$  and  $f_1 \perp t_{n-1}$  for arbitrary  $t_{n-1} \in \mathcal{T}_{2n-1}$ . Therefore

$$\int_{-\pi}^{\pi} (f_1(t) - t_{n-1}(t)) \cos ntdt = \int_{-\pi}^{\pi} f_1(t) \cos ntdt = \pi\psi(n) \quad \forall t_{n-1} \in \mathcal{T}_{2n-1}. \quad (16)$$

On the other hand, taking into account the proposition 2 for  $q = 1$ ,  $p = s$ , we get

$$\begin{aligned} \int_{-\pi}^{\pi} (f_1(t) - t_{n-1}(t)) \cos ntdt &\leq \|f_1 - t_{n-1}\|_1 \leq \\ &\leq (2\pi)^{1-\frac{1}{s}} \|f_1 - t_{n-1}\|_s, \quad 1 \leq s \leq \infty, \quad \forall t_{n-1} \in \mathcal{T}_{2n-1}. \end{aligned} \quad (17)$$

In view of (16)–(17) we arrive at the inequalities

$$E_n(L_{\beta,\infty}^\psi)_s \geq E_n(f_1)_s = \inf_{t_{n-1} \in \mathcal{T}_{2n-1}} \|f_1 - t_{n-1}\|_s \geq \frac{1}{2}\psi(n), \quad 1 \leq s \leq \infty. \quad (18)$$

Theorem 1 is proved.

We denote by  $B$  the set of positive sequences  $\psi(k)$ ,  $k \in \mathbb{N}$ , for each of which there exists a positive constant  $K$  such that  $\frac{\psi(k)}{\psi(2k)} \leq K$ ,  $k \in \mathbb{N}$ . The sequences  $\psi(k) = k^{-r}$ ,  $r > 0$ ,  $\psi(k) = \ln^{-\varepsilon}(k+1)$ ,  $\varepsilon > 0$ , etc. are representatives of the set  $B$ .

**Theorem 2.** *Let  $\psi \in P \cap B$ ,  $1 \leq s < \infty$  and  $\beta \in \mathbb{R}$ . Then*

$$e_{2n}^\perp(L_{\beta,\infty}^\psi)_s \asymp e_{2n-1}^\perp(L_{\beta,\infty}^\psi)_s \asymp \psi(n). \quad (19)$$

*Proof.* It follows from the formulas (7) and (9), that under the conditions of the theorem 1, next inequalities are true

$$e_{2n}^\perp(L_{\beta,\infty}^\psi)_s \leq e_{2n-1}^\perp(L_{\beta,\infty}^\psi)_s \leq \mathcal{E}_n(L_{\beta,\infty}^\psi)_s \leq K^{(1)}\psi(n). \quad (20)$$

Now we determine a lower bound of the quantity  $e_{2n}^\perp(L_{\beta,\infty}^\psi)_s$ . For this we use the well-known result of Rudin–Shapiro (see, e.g., lemma 6.32.1 in [21]).

**Proposition 3.** *There exists sequence of numbers  $\{\varepsilon_k\}_{k=0}^\infty$ , such that  $\varepsilon_k = \pm 1$  and*

$$\left\| \sum_{k=0}^m \varepsilon_k e^{ikx} \right\|_\infty \leq 5\sqrt{m+1}, \quad m = 0, 1, \dots \quad (21)$$

Taking into account proposition 3 for  $m = 2n - 1$ , we choose the sequence of numbers  $\{\xi_k\}_{k=0}^\infty$ ,  $\xi_k = \pm 1$  such that

$$\left\| \sum_{k=0}^{2n-1} \xi_k e^{ikx} \right\|_\infty \leq 5\sqrt{2n}. \quad (22)$$

We set

$$\psi(0) := \psi(1)$$

and consider the function

$$f_2(t) = f_2(\psi; n; t) := \frac{1}{10\sqrt{2n} + 2} \sum_{k=-2n+1}^{2n-1} \xi_{|k|} \psi(|k|) e^{ikt}. \quad (23)$$

Since, according to definition of  $(\psi, \beta)$ -derivative and the inequality (22),

$$\begin{aligned} \|(f_2)_\beta^\psi\|_\infty &= \frac{1}{10\sqrt{2n} + 2} \left\| \sum_{k=1}^{2n-1} \xi_k e^{i(k t + \frac{\beta\pi}{2})} + \sum_{k=1}^{2n-1} \xi_k e^{i(-k t - \frac{\beta\pi}{2})} \right\|_\infty \leq \\ &\leq \frac{1}{10\sqrt{2n} + 2} \left( \left\| \sum_{k=1}^{2n-1} \xi_k e^{i(k t + \frac{\beta\pi}{2})} \right\|_\infty + \left\| \sum_{k=1}^{2n-1} \xi_k e^{i(-k t - \frac{\beta\pi}{2})} \right\|_\infty \right) = \\ &= \frac{1}{5\sqrt{2n} + 1} \left\| \sum_{k=1}^{2n-1} \xi_k e^{ikt} \right\|_\infty \leq 1, \end{aligned}$$

so  $f_2 \in L_{\beta,\infty}^\psi$ .

We consider the quantity

$$I = \inf_{\gamma_{2n}} \left| \int_{-\pi}^{\pi} (f_2(t) - S_{\gamma_{2n}}(f_2; t)) \sum_{k=-2n+1}^{2n-1} \xi_{|k|} e^{ikt} dt \right|.$$

By virtue of Holder's inequality, proposition 2 and correlation (22) for  $1 \leq s < \infty$ ,  $\frac{1}{s} + \frac{1}{s'} = 1$

$$\begin{aligned} I &\leq \inf_{\gamma_{2n}} \|f_2(t) - S_{\gamma_{2n}}(f_2; t)\|_s \left\| \sum_{k=-2n+1}^{2n-1} \xi_{|k|} e^{ikt} \right\|_{s'} = \\ &= e_{2n}^\perp(f_2)_s \left\| \sum_{k=-2n+1}^{2n-1} \xi_{|k|} e^{ikt} \right\|_{s'} \leq (2\pi)^{\frac{1}{s'}} e_{2n}^\perp(f_2)_s \left\| \sum_{k=-2n+1}^{2n-1} \xi_{|k|} e^{ikt} \right\|_\infty \leq \end{aligned}$$

$$\begin{aligned}
&\leq 2\pi e_{2n}^\perp(f_2)_s \left( \left\| \sum_{k=0}^{2n-1} \xi_k e^{ikt} \right\|_\infty + \left\| \sum_{k=1}^{2n-1} \xi_k e^{-ikt} \right\|_\infty \right) \leq \\
&\leq 2\pi e_{2n}^\perp(f_2)_s \left( 2 \left\| \sum_{k=0}^{2n-1} \xi_k e^{ikt} \right\|_\infty + 1 \right) \leq 2\pi(10\sqrt{2n} + 1) e_{2n}^\perp(f_2)_s.
\end{aligned} \tag{24}$$

On the other hand, taking into account the orthogonality of trigonometric system  $\{e^{ikt}\}$  and the fact that  $\xi_k^2 = 1$ , we obtain

$$\begin{aligned}
I &= \frac{1}{10\sqrt{2n} + 2} \inf_{\gamma_{2n}} \left| \int_{-\pi}^{\pi} \sum_{\substack{|k| \leq 2n-1, \\ k \notin \gamma_{2n}}} \xi_{|k|} \psi(|k|) e^{ikt} \sum_{k=-2n+1}^{2n-1} \xi_{|k|} e^{ikt} dt \right| = \\
&= \frac{\pi}{5\sqrt{2n} + 1} \inf_{\gamma_{2n}} \sum_{\substack{|k| \leq 2n-1, \\ k \notin \gamma_{2n}}} \psi(|k|).
\end{aligned} \tag{25}$$

Since the sequence  $\psi(k)$  almost decreases, so

$$\inf_{\gamma_{2n}} \sum_{\substack{|k| \leq 2n-1, \\ k \notin \gamma_{2n}}} \psi(|k|) \geq K^{(2)} \inf_{\gamma_{2n}} \sum_{\substack{|k| \leq 2n-1, \\ k \notin \gamma_{2n}}} \psi(2n-1) = K^{(2)} \psi(2n-1)(2n-1). \tag{26}$$

In view of (24)–(26) we get

$$e_{2n}^\perp(f_2)_s \geq \frac{K^{(2)} \psi(2n-1)(2n-1)}{(10\sqrt{2n} + 2)(10\sqrt{2n} + 1)} \geq K^{(3)} \psi(2n). \tag{27}$$

Since, if  $\psi \in B$ , so  $\psi(2n) \geq K^{(4)} \psi(n)$ , and, hence, taking into account (27), we find

$$e_{2n}^\perp(L_{\beta,\infty}^\psi)_s \geq e_{2n}^\perp(f_2)_s \geq K^{(5)} \psi(n). \tag{28}$$

Estimates (19) follow from (20) and (28). Theorem 2 is proved.

**Corollary 1.** *Let  $r > 0$ ,  $1 \leq s < \infty$  and  $\beta \in \mathbb{R}$ . Then*

$$e_{2n}^\perp(W_{\beta,\infty}^r)_s \asymp e_{2n-1}^\perp(W_{\beta,\infty}^r)_s \asymp n^{-r}. \tag{29}$$

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